Brownian motion and Stochastic Calculus
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Assignment 6—solutions

Exercise 1

Let $(W_t)_{t\geq 0}$ be a Brownian motion. For any $a>0$ consider the random times

$$T_a := \inf\big\{t>0 : W_t \geq a\big\},\; \overline{T}_a := \inf\big\{t>0 : |W_t| \geq a\big\},$$

1) Show that these random times are $\mathbb{F}^{W,\mathbb{P}}$–stopping times.

2) Show that the Laplace transform of T_a has the value

$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\mu T_a)\big] = \exp\big(-a\sqrt{2\mu}\big),\; \forall \mu>0,$$

and show that $\mathbb{P}[T_a<\infty]=1$.

Hint: consider the martingale $M_t^\lambda := \exp(\lambda W_t - \frac{\lambda^2}{2}t)$ and use the optional sampling theorem.

4) Show that the Laplace transform of $\overline{T}_a$ has the value

$$\mathbb{E}^{\mathbb{P}}\big[\exp(-\mu \overline{T}_a)\big] = \frac{1}{\cosh(a\sqrt{2\mu})},\; \forall \mu>0.$$

1) **These are the first entry times of closed sets by an $\mathbb{F}$-adapted and continuous process, making them $\mathbb{F}$–stopping times.**

2) **Notice first that we know that $\mathbb{P}[T_a<+\infty]=1$ since $\limsup_{t\to+\infty} W_t = -\liminf_{t\to\infty} W_t = +\infty$. Next, for any $n\in\mathbb{N}$, we define**

$$T_a^n := T_a \wedge n.$$

Since T_a^n is a bounded $\mathbb{F}$–stopping time, the optional stopping theorem implies that for $n\in\mathbb{N}$ and any $\lambda\in\mathbb{R}$

$$\mathbb{E}^{\mathbb{P}}[M^\lambda_{T_a^n}] = M_0^\lambda = 1.$$

Moreover, on the event $\{T_a<\infty\}$ we have

$$\exp\bigg(\lambda W_{T_a^n} - \frac{\lambda^2}{2}(T_a^n)\bigg) \overset{n\to\infty}{\longrightarrow} \exp\bigg(\lambda W_{T_a} - \frac{\lambda^2}{2}T_a\bigg) = \mathrm{e}^{\lambda a}\exp\bigg(-\frac{\lambda^2}{2}T_a\bigg).$$

We conclude that for any $\lambda>0$

$$\exp\bigg(\lambda W_{T_a^n} - \frac{\lambda^2}{2}(T_a^n n)\bigg) \overset{n\to\infty}{\longrightarrow} \mathrm{e}^{\lambda a}\exp\bigg(-\frac{\lambda^2}{2}T_a\bigg),\; \mathbb{P}\text{–a.s.} \tag{0.1}$$

Observe that for any $n\in\mathbb{N}$ we have

$$0\leq \exp\bigg(\lambda W_{T_a^n} - \frac{\lambda^2}{2}(T_a^n)\bigg) \leq \mathrm{e}^{\lambda a}.$$

Thus, we deduce from (0.1), by applying the dominated convergence theorem, that for any $\lambda>0$

$$1 = \mathbb{E}^{\mathbb{P}}\big[M^\lambda_{T_a^n}\big] \overset{n\to\infty}{\longrightarrow} \mathrm{e}^{\lambda a}\mathbb{E}^{\mathbb{P}}\bigg[\exp\bigg(-\frac{\lambda^2}{2}T_a\bigg)\bigg],$$

and so, for any $\lambda > 0$

$$\mathrm{e}^{\lambda a}\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\frac{\lambda^2}{2}T_a\right)\right] = 1. \tag{0.2}$$

Fix any $\mu > 0$. For $\lambda := \sqrt{2\mu}$, (0.2) yields the desired result.

3) **For any $\lambda > 0$, consider the martingale $(N_t^\lambda)_{t\geq 0}$ defined by**

$$N_t^\lambda := \frac{M_t^\lambda + M_t^{-\lambda}}{2} = \cosh\big(\lambda W_t\big)\exp\left(-\frac{\lambda^2}{2}t\right) = \cosh\big(\lambda|W_t|\big)\exp\left(-\frac{\lambda^2}{2}t\right).$$

The procedure in 1) (using now N^λ instead of M^λ and $\overline{T}_a$ instead of T_a), using the inequality $0 \leq N_{\overline{T}_a\wedge n} \leq \cosh(\lambda a)$, yields

$$\cosh(\lambda a)\mathbb{E}^{\mathbb{P}}\left[\exp\left(-\frac{\lambda^2}{2}\overline{T}_a\right)\right] = 1. \tag{0.3}$$

Fix any $\mu > 0$. For $\lambda := \sqrt{2\mu}$, (0.3) yields the desired result.

Exercise 2

Let W be a Brownian motion on $[0,\infty)$ and $S_0 > 0$, $\sigma > 0$, $\mu \in \mathbb{R}$ constants. The stochastic process $S = (S_t)_{t\geq 0}$ given by

$$S_t := S_0\exp\big(\sigma W_t + (\mu - \sigma^2/2)t\big),$$

is called *geometric Brownian motion.*

1) Prove that for $\mu \neq \sigma^2/2$, we have

$$\lim_{t\to\infty} S_t = +\infty,\ \mathbb{P}\text{–a.s., or } \lim_{t\to\infty} S_t = 0,\ \mathbb{P}\text{–a.s.}$$

   When do the respective cases arise?

2) Discuss the behaviour of S_t as $t \longrightarrow \infty$ in the case $\mu = \sigma^2/2$.

3) For $\mu = 0$, show that S is a martingale, but not uniformly integrable.

1) **From the definition of S_t, we get $\frac{1}{t}\log\frac{S_t}{S_0} = \sigma\frac{W_t}{t} + \mu - \frac{1}{2}\sigma^2$. The strong law of large numbers then gives, $\mathbb{P}$–a.s.,**

$$\lim_{t\to\infty}\log\frac{S_t}{S_0} = \begin{cases} +\infty, & \textbf{if } \mu - \frac{1}{2}\sigma^2 > 0, \\ -\infty, & \textbf{if } \mu - \frac{1}{2}\sigma^2 < 0, \end{cases}$$

and therefore, $\mathbb{P}$–a.s.,

$$\lim_{t\to\infty} S_t = \begin{cases} +\infty, & \textbf{if } \mu - \frac{1}{2}\sigma^2 > 0, \\ 0, & \textbf{if } \mu - \frac{1}{2}\sigma^2 < 0. \end{cases}$$

2) **If $\mu = \frac{1}{2}\sigma^2$, then $S_t = S_0\mathrm{e}^{\sigma W_t}$. From the law of the iterated logarithm we have**

$$\limsup_{t\to\infty}\frac{W_t}{\sqrt{2t\log\log t}} = 1,\ \mathbb{P}\text{–a.s.},\ \liminf_{t\to\infty}\frac{W_t}{\sqrt{2t\log\log t}} = -1,\ \mathbb{P}\text{–a.s.}$$

As a consequence, $\mathbb{P}$–almost every path $W_\cdot(\omega)$ oscillates between $+\infty$ and $-\infty$ for $t \to \infty$. Therefore, S_t oscillates between 0 and $+\infty$ for $t \to \infty$.

3) **For $\mu = 0$, it is known that S is a martingale. Moreover, by 1), we have that**

$$S_t \overset{t\to\infty}{\longrightarrow} 0,\ \mathbb{P}\text{–a.s.}$$

As a martingale with $S_0 > 0$, S cannot converge to 0 in $\mathbb{L}^1(\mathbb{R},\mathcal{F},\mathbb{P})$. Thus, S is not $\mathbb{P}$–uniformly integrable.

Exercise 3

Let B be a standard Brownian motion. Let $S^\star \in [0,1]$ be the smallest $s \in [0,1]$ with $B_s = \sup_{t\in[0,1]} B_t$. Moreover, let $L := \sup\{t \in [0,1] \colon B_t = 0\}$ be the last time in the interval $[0,1]$ when B is at 0.

1) Show that $\mathbb{P}$–a.s., B attains its maximum on the interval $[0,1]$ at a unique point.

2) Let W be a standard Brownian motion, independent of B. Prove that whenever $s \in [0,1]$, we have

$$\mathbb{P}[S^\star < s] = \mathbb{P}\bigg[\sup_{t\in[0,s]} B_t > \sup_{t\in[0,1-s]} W_t\bigg].$$

3) Let N and N' be random variables distributed as $N(0,1)$ and independent. Show that

$$\mathbb{P}[S^\star < s] = \mathbb{P}\big[\sqrt{s}|N| > \sqrt{1-s}|N'|\big] = 2\arcsin(\sqrt{s})/\pi.$$

The law of $S^\star$ is called the Arcsine distribution.

4) Show also that

$$\mathbb{P}[L < s] = \mathbb{P}\bigg[\sup_{t\in[0,s]} B_t > \sup_{t\in[0,1-s]} W_t\bigg], \text{ for } s \in [0,1],$$

so that L and $S^\star$ have the same law.

In this question, we will use the following observation. For $s \in [0,1]$

$$\sup_{t\in[0,s]}\{B_t - B_s\} = \sup_{t\in[0,s]}\{B_{s-t} - B_s\} \overset{\text{law}}{=} \sup_{t\in[0,s]} B_t \overset{\text{law}}{=} |B_s| \overset{\text{law}}{=} \sqrt{s}|N|,$$
$$\sup_{t\in[s,1]}\{B_t - B_s\} = \sup_{t\in[0,1-s]}\{B_{s+t} - B_s\} \overset{\text{law}}{=} \sup_{t\in[0,1-s]} B'_t \overset{\text{law}}{=} |B'_{1-s}| \overset{\text{law}}{=} \sqrt{1-s}|N'|,$$

where N and N' are $\mathbb{P}$-independent standard Gaussian random variables, and B' is a Brownian motion $\mathbb{P}$-independent of B. Note also that in both lines we used the weak Markov property (plus time inversion in the first line) and then the reflection principle. Also, by the weak Markov property at time s, the random variables $\sup_{t\in[0,s]}\{B_t - B_s\}$ and $\sup_{t\in[s,1]}\{B_t - B_s\}$ are $\mathbb{P}$-independent and hence

$$\bigg(\sup_{t\in[0,s]}\{B_t - B_s\}, \sup_{t\in[s,1]}\{B_t - B_s\}\bigg) \overset{\text{law}}{=} \bigg(\sup_{t\in[0,s]} B_t, \sup_{t\in[0,1-s]} B'_t\bigg) \overset{\text{law}}{=} (\sqrt{s}|N|, \sqrt{1-s}|N'|).$$

1) **It is easy to see that it suffices to prove that for each $s \in \mathbb{Q}\cap(0,1)$, $\mathbb{P}$–almost surely $\sup_{t\in[0,s]} B_t \neq \sup_{t\in[s,1]} B_t$. This follows from the observations made above, indeed, we get**

$$\mathbb{P}\bigg[\sup_{t\in[0,s]} B_t \neq \sup_{t\in[s,1]} B_t\bigg] = \mathbb{P}\bigg[\sup_{t\in[0,s]}\{B_t - B_s\} \neq \sup_{t\in[s,1]}\{B_t - B_s\}\bigg] = \mathbb{P}\big[\sqrt{s}|N| \neq \sqrt{1-s}|N'|\big] = 1.$$

2) **Suppose that $s \in [0,1]$, then using the observations made above**

$$\mathbb{P}[S^\star < s] = \mathbb{P}\bigg[\sup_{t\in[0,s]} B_t > \sup_{t\in[s,1]} B_t\bigg] = \mathbb{P}\bigg[\sup_{t\in[0,s]}\{B_t - B_s\} > \sup_{t\in[s,1]}\{B_t - B_s\}\bigg] = \mathbb{P}\bigg[\sup_{t\in[0,s]} B_t > \sup_{t\in[0,1-s]} B'_t\bigg].$$

3) **Using the final equality in law derived above and the isotropy of the law of (N,N') we finally obtain**

$$\begin{aligned}\mathbb{P}[S^\star < s] &= \mathbb{P}\bigg[\sup_{t\in[0,s]} B_t > \sup_{t\in[0,1-s]} B'_t\bigg] = \mathbb{P}\big[\sqrt{s}|N| > \sqrt{1-s}|N'|\big]\\ &= \mathbb{P}\big[(N,N') \in \{(\pm r\cos(\theta), r\sin(\theta))\colon |\theta| < \arcsin(\sqrt{s}),\ r>0\}\big]\\ &= 4\arcsin(\sqrt{s})/(2\pi) = 2\arcsin(\sqrt{s})/\pi.\end{aligned}$$

4) **For $s \in [0,1]$, we observe**

$$\{L < s\} = \bigg\{B_s < 0, \sup_{t\in[s,1]}\{B_t - B_s\} < |B_s|\bigg\} \bigcup \bigg\{-B_s < 0, \sup_{t\in[s,1]}\{(-B_t) - (-B_s)\} < |B_s|\bigg\}.$$

The two events in the union are disjoint, and since B and $-B$ have the same law, we can therefore compute

$$\mathbb{P}[L < s] = 2\mathbb{P}\Big[B_s < 0,\ \sup_{t\in[s,1]}\{B_t - B_s\} < |B_s|\Big] = \mathbb{P}\Big[\sup_{t\in[s,1]}\{B_t - B_s\} < |B_s|\Big].$$

We observe that by the weak Markov property $|B_s|$ and $\sup_{t\in[s,1]}\{B_t - B_s\}$ are $\mathbb{P}$-independent. Also $|B_s| \overset{\text{law}}{=} \sup_{t\in[0,s]} B_t$ and $\sup_{t\in[s,1]}\{B_t - B_s\} \overset{\text{law}}{=} \sup_{t\in[0,1-s]} B'$ and since B and B' are $\mathbb{P}$-independent by assumption, we get

$$\Big(|B_s|,\ \sup_{t\in[s,1]}\{B_t - B_s\}\Big) \overset{\text{law}}{=} \Big(\sup_{t\in[0,s]} B_t,\ \sup_{t\in[0,1-s]} B'_t\Big),$$

from which the result is immediate.

Exercise 4

Let $(B_t)_{t\geq 0}$ be a Brownian motion and $M_t := \sup_{s\leq t} B_s$. Show that the joint distribution of the pair (B_t, M_t) is absolutely continuous with density

$$f_t(x,y) := \frac{2(2y-x)}{\sqrt{2\pi t^3}} \exp\Big(-\frac{(2y-x)^2}{2t}\Big)\mathbf{1}_{\{y\geq 0\}}\mathbf{1}_{\{x\leq y\}},\ (x,y)\in\mathbb{R}^2.$$

Hint: Show that

(i) for $y > 0$, $x \leq y$, $\mathbb{P}[B_t \leq x,\ M_t \geq y] = \mathbb{P}[B_t \geq 2y - x]$;

(ii) for $y > 0$, $x \leq y$,

$$\mathbb{P}[B_t \leq x,\ M_t \leq y] = \Phi\Big(\frac{x}{\sqrt{t}}\Big) - \Phi\Big(\frac{x-2y}{\sqrt{t}}\Big),$$

where Φ is the distribution function of a standard Gaussian random variable;

(iii) for $y > 0$, $x \geq y$,

$$\mathbb{P}[B_t \leq x,\ M_t \leq y] = \mathbb{P}[M_t \leq y] = \Phi\Big(\frac{y}{\sqrt{t}}\Big) - \Phi\Big(-\frac{y}{\sqrt{t}}\Big),$$

and for $y \leq 0$, $\mathbb{P}[B_t \leq x,\ M_t \leq y] = 0$.

To show the reflection principle, let $T_y = \inf\{t > 0 : B_t \geq y\}$ be the first time the Brownian motion is greater than y. Then, $\{T_y \leq t\} = \{M_t \geq y\}$ for $y \geq 0$. Furthermore, since $B_{T_y} = y$, we have

$$\mathbb{P}\big[B_t \leq x, M_t \geq y\big] = \mathbb{P}\big[B_t \leq x, T_y \leq t\big] = \mathbb{P}\big[B_t - B_{T_y} \leq x - y, T_y \leq t\big].$$

Relying on the strong Markov property, we obtain

$$\mathbb{P}\big[B_t - B_{T_y} \leq x-y, T_y \leq t\big] = \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{T_y\leq t\}}\mathbb{P}[B_t - B_{T_y} \leq x - y|T_y]\big] = \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{T_y\leq t\}}\mathbb{P}[B_t - B_{T_y} \leq x-y]\big],$$

since $(\tilde{B}_u := B_{T_y+u} - B_{T_y}, u \geq 0)$ is a Brownian motion independent of $(B_t, t \leq T_y)$. We also note that $-\tilde{B}$ and $\tilde{B}$ have the same law. Hence,

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{T_y\leq t\}}\mathbb{P}[B_t - B_{T_y} \leq x-y]\big] &= \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{T_y\leq t\}}\mathbb{P}[B_t - B_{T_y} \geq y-x]\big] \\ &= \mathbb{E}^{\mathbb{P}}\big[\mathbf{1}_{\{T_y\leq t\}}\mathbb{P}[B_t - B_{T_y} \geq y-x|T_y]\big] \\ &= \mathbb{P}[B_t \geq 2y - x, T_y \leq t]. \end{aligned} \tag{0.4}$$

The right-hand side of (0.4) is equal to $\mathbb{P}[B_t \geq 2y - x]$ **since, from** $x \leq y$ **we have** $2y - x \geq y$ **which implies that, on the set** $\{B_t \geq 2y - x\}$**, one has** $M_t \geq y$**. Therefore, it follows that, for** $y \geq 0, x \leq y$,

$$\mathbb{P}[B_t \leq x, M_t \leq y] = \mathbb{P}[B_t \leq x] - \mathbb{P}[B_t \leq x, M_t \geq y] \overset{(1)}{=} \mathbb{P}[B_t \leq x] - \mathbb{P}[B_t \geq 2y - x], \tag{0.5}$$

and hence the first hint is obtained.

For $0 \leq y \leq x$, **since** $M_t \geq B_t$ **we get**

$$\mathbb{P}[B_t \leq x, M_t \leq y] = \mathbb{P}[B_t \leq y, M_t \leq y] = \mathbb{P}[M_t \leq y].$$

Furthermore, by setting $x = y$ **in** (0.5)

$$\mathbb{P}[B_t \leq y, M_t \leq y] = \Phi\left(\frac{y}{\sqrt{t}}\right) - \Phi\left(-\frac{y}{\sqrt{t}}\right),$$

hence the second hint is obtained. Finally, noticed that for $y < 0$,

$$\mathbb{P}[B_t \leq x, M_t \leq y] = 0.$$

since $M_t \geq M_0 = 0$. **Finally, the density for the joint law of** (B, M) **is obtained by taking derivatives.**